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LETTER TO THE EDITOR

Invariants for a cubic three-wave system

L Cairó, D D Hua and M R Feix
MAPM/CNRS-Université d'Orléans, 45071 Orleans, Cedex 2, France

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Abstract. Invariants of the motion (in general time-dependent first integrals) for a cubic three-wave system are obtained using the generalized Carleman method. It is shown that a subset of these invariants are of Lotka–Volterra type.

Recently there has been much research on partial integrability of three-dimensional dynamical systems by searching for their invariants of the motion (first integrals), from predictive methods such as the singularity analyses [1, 2] and from direct methods such as the Carleman methods [3, 4], and others [5–10]. However, despite all the successes of these methods most of the systems so far studied have been quadratic. Hence here we use a generalization of the Carleman method, introduced previously [4], to search invariants for the modes of a three-wave interaction system [11, 12], which is a three-dimensional cubic system. There are two purposes for doing so; one is the physical interest in the regular (non-chaotic) motion of three-interacting waves which has applications in the theory of optical maser [13] and in the study of gravity-wave interactions [14]. The other is to find explicit conditions for the existence of invariants which would serve as a guide for future extension of the Painlevé, singularity analyses and other methods. Indeed, the Carleman method with the assistance of computer algebra facilities is still the most general and successful method in finding quasi-polynomial invariants, which are the only known type for all three-dimensional systems [3, 4]. For example, Kuŝ [3], using the Carleman method found all six known Lorenz invariants and the conditions for their existence. The application of Painlevé analysis to this system came only later with the work of Levine and Tabor [15] who, using an appropriate singularity analysis, suggest that no other invariants may exist for the Lorenz system. Furthermore there exist partially integrable systems that do not satisfy the Painlevé property [15, 16]. The system under consideration is the so-called diagonalized system, where the linear terms appear in a diagonalized form and the cubic terms in a 'factorized form' [6]. The system is defined as

$$\dot{x}_i = x_i\left(\lambda_i + \sum_{j=1}^{3} N_{ij}x_j^2\right) + \gamma_i \prod_{j\neq i} x_j \tag{1}$$

(here i goes from 1–3). Note that the particular case $N_{ij} = 0$ and $\lambda_i = 0\ \forall i, j$ is the well known Euler system, the solution of which can be exhibited in terms of elliptic functions. See, for example, Landau and Lifchitz [17]. Also note that when $\gamma_i = 0\ \forall i$, the system becomes Lotka–Volterra (LVS) as it can be seen that the transformation $\bar{x}_i = x_i^2$, changes (1) into

$$\dot{\bar{x}}_i = \bar{x}_i\left(\bar{\lambda}_i + \sum_{j=1}^{3} \bar{N}_{ij}\bar{x}_j\right) = 0$$

where $\bar{\lambda_i} = 2\lambda_i$ and $\bar{N_{ij}} = 2N_{ij}$.

We recall that the generalized Carleman method [4] consists of the following ansatz for the quasi-polynomial invariant:

$$I = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \sum_{l,m,n=0}^{L,M,N} A_{lmn} x_1^l x_2^m x_3^n \, e^{st} \tag{2}$$

where l, m, n are integers and α_1, α_2 and α_3 are real numbers. We will limit our investigations to the cases where $l + m + n \leqslant P$ with $P = 2$ i.e. L, M, and $N \leqslant 2$. We have, altogether, 10 coefficients $A_{l,m,n}$ and three arbitrary parameters α, β, γ. We write the invariance of (2)

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_1}\dot{x_1} + \frac{\partial I}{\partial x_2}\dot{x_2} + \frac{\partial I}{\partial x_3}\dot{x_3} = 0$$

where we introduce the expressions for $\dot{x_1}, \dot{x_2}, \dot{x_3}$ given by (1). The partial derivative with respect to the time of the term $A_{lmn} x_1^{\alpha_1 + l} x_2^{\alpha_2 + m} x_3^{\alpha_3 + n}$ brings the same term multiplied by s. The derivative with respect x_1 brings $(\alpha + l) A_{lmn} x_1^{\alpha + l - 1} x_2^{\beta + m} x_3^{\gamma + n} \dot{x_1}$, i.e. omiting $x_1^\alpha x_2^\beta x_3^\gamma$, we get terms in the following monomials:

$$x_1^l x_2^m x_3^n \qquad x_1^{l+2} x_2^m x_3^n \qquad x_1^l x_2^{m+2} x_3^n \qquad x_1^l x_2^m x_3^{n+2} \qquad x_1^{l-1} x_2^{m+1} x_3^{n+1} \, .$$

We use Maple to compute, gather terms of identical power in lmn and solve the equations resulting from setting all the coefficients equal to zero (see [4] for further details of the method). The equations are linear in the coefficients A_{lmn} and can be written in the form

$$m_{ijk} = \sum_l \sum_m \sum_n [ijklmn] A_{lmn} = 0$$

where the ijk and lmn refers, respectively, to the columns and lines of the matrix appearing in table 1 (the so-called Carleman matrix). Each $[ijklmn]$ is one of the seven multiplying operators acting on A_{lmn} (denoted by $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$) given by

$$\begin{aligned}
\mathbf{0} &= (l+\alpha)\lambda_1 + (m+\beta)\lambda_2 + (n+\gamma)\lambda_3 + s \\
\mathbf{1} &= (l+\alpha)N_{11} + (m+\beta)N_{21} + (n+\gamma)N_{31} \\
\mathbf{2} &= (l+\alpha)N_{12} + (m+\beta)N_{22} + (n+\gamma)N_{32} \\
\mathbf{3} &= (l+\alpha)N_{13} + (m+\beta)N_{23} + (n+\gamma)N_{33} \\
\mathbf{4} &= (n+\gamma)\gamma_3 \qquad \mathbf{5} = (m+\beta)\gamma_2 \qquad \mathbf{6} = (l+\alpha)\gamma_1 \, .
\end{aligned}$$

Many of the equations are consequently redundant and we end up with a reasonable amount of constraints on the coefficients of (1). The invariant conditions and the corresponding invariants found are the following.

(i) $\gamma_i = 0 \;\; \forall i$ and $\det N_{ij} = 0$:

$$I = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} e^{st} \tag{3}$$

where $s = -\alpha_i \lambda_i$ and α_is are solutions to $\alpha_i N_{ij} = 0$ (from here on the Einstein summation rule on repeated indexes is used). This is the invariant I in [4].

(ii) $\gamma_i = 0$, $\lambda_i = \lambda \;\; \forall i$ and $R_{123} \equiv d_{12} d_{23} d_{31} + d_{21} d_{32} d_{13} = 0$ where $d_{ij} = N_{ij} - N_{jj} \;\; \forall i, j$:

$$I = x_1^{2\alpha_1} x_2^{2\alpha_2} x_3^{2\alpha_3} \left(x_1^2 - \frac{d_{12}}{d_{21}} x_2^2 - \frac{d_{13}}{d_{31}} x_3^2 \right) \tag{4}$$

where the α_is are the solutions of the system $\alpha_i N_{ij} = -N_{jj}$ $(i, j = 1, \ldots, 3)$. This is the invariant II in [4].

Table 1. Carleman matrix.

	000	−111	001	010	1−11	100	11−1	−112	−121	002	011	020	1−12	101	110	12−1	2−11	200	21−1
000	**0**	**6**			**5**		**4**			**3**		**2**						**1**	
001			**0**					**6**					**5**		**4**				
010				**0**					**6**					**5**		**4**			
100						**0**					**6**						**5**		**4**
002										**0**									
011											**0**								
020												**0**							
101														**0**					
110															**0**				
200																		**0**	

	−113	−122	−131	003	012	021	030	1−13	102	111	120	13−1	2−12	201	210	22−1	3−11	300	31−1
001				**3**		**2**								**1**					
010					**3**		**2**								**1**				
100									**3**		**2**							**1**	
002	**6**							**5**		**4**									
011		**6**							**5**		**4**								
020			**6**							**5**		**4**							
101					**6**								**5**		**4**				
110						**6**								**5**		**4**			
200										**6**							**5**		**4**

	004	013	022	031	040	103	112	121	130	202	211	220	301	310	400
002	**3**		**2**							**1**					
011		**3**		**2**							**1**				
020			**3**		**2**							**1**			
101						**3**		**2**					**1**		
110							**3**		**2**					**1**	
200										**3**		**2**			**1**

(iii) $\gamma_i = 0 \;\; \forall i$, $\lambda_1 = \lambda_3 = \lambda$ and $N_{32} = N_{12}$:

$$I = x_1^{2\alpha_1}\, x_2^{2\alpha_2}\, x_3^{2\alpha_3} \left(x_1^2 - \frac{d_{13}}{d_{31}} x_3^2 \right) \mathrm{e}^{st} \tag{5}$$

where α_is are once again solutions to $\alpha_i N_{ij} = -N_{jj}$ and $s = -2\lambda\,(1 + \alpha_1 + \alpha_3) - 2\alpha_2\lambda_2$. This is invariant II′ in [4]. Note that we obtain two more invariants of this form by exchanging the indices $(1 \to 2,\, 2 \to 1)$ and $(3 \to 2,\, 2 \to 3)$.

(iv) $\gamma_i = 0$ and $R_{ij} \equiv (N_{ii}/\lambda_i)\left(N_{ij} - N_{jj}\right) + (N_{jj}/\lambda_j)\left(N_{jj} - N_{jj}\right) = 0 \;\; \forall i, j$:

$$I = x_1^{2\alpha_1}\, x_2^{2\alpha_2}\, x_3^{2\alpha_3} \left(1 + \frac{N_{11}}{\lambda_1} x_1^2 + \frac{N_{22}}{\lambda_2} x_2^2 + \frac{N_{33}}{\lambda_3} x_3^2 \right) \mathrm{e}^{st} \tag{6}$$

where the α_is are once again solutions to $\alpha_i N_{ij} = -N_{jj}$ and $s = -2\alpha_i\lambda_i$. This is known as invariant III in [4].

(v) $\gamma_2 = \gamma_3 = 0$ and $N_{33}/N_{23} = N_{32}/N_{22} = N_{31}/N_{21}$:

$$I = x_2^{-N_{31}}\, x_3^{N_{21}}\, \mathrm{e}^{(N_{31}\lambda_2 - N_{21}\lambda_3)t}\,. \tag{7}$$

(vi) $\gamma_2 = \gamma_3 = 0$ and $N_{1i} = N_{2i} + N_{3i}$ $\forall i$:

$$I = x_2^{-1} x_3^{-1} \left(x_1 + \frac{\gamma_1}{\lambda_1 - \lambda_2 - \lambda_3} x_2 x_3 \right) e^{(-\lambda_1+\lambda_2+\lambda_3)t} . \tag{8}$$

(vii) $\gamma_2 = \gamma_3 = 0$, $\lambda_1 = \lambda_3 - \lambda_2$, $N_{13}N_{31} = N_{11}N_{33}$, $N_{21}N_{33} = N_{31}N_{23}$ and $N_{13} = N_{33} - N_{23}$:

$$I = x_2^{\alpha_2} x_3^{\alpha_3} \left(x_3 + \frac{N_{12} + N_{22} - N_{32}}{\gamma_1} x_1 x_2 \right) e^{st} \tag{9}$$

with $\alpha_2 = (N_{32} - N_{12} - N_{22})N_{31}/(N_{22}N_{31} - N_{32}N_{21})$, $\alpha_3 = (N_{12}N_{21} - N_{11}N_{22})/(N_{22}N_{31} - N_{32}N_{21})$, $s = -\lambda_2\alpha_2 - \lambda_3(1 + \alpha_3)$.

Note that a similar invariant to (9) exists changing indexes ($2 \to 3$ and $3 \to 2$) in the preceding formulation.

(viii) $\gamma_2 = \gamma_3 = 0$, $N_{12}N_{21} = N_{11}N_{22}$, $N_{32}N_{23} = N_{22}N_{33}$, $N_{13}N_{31} = N_{33}N_{11}$, $N_{21} = N_{11} - N_{31}$ and $N_{23} = N_{13} - N_{33}$:

$$I = x_3^{\alpha_3} \left(x_1 + \frac{\gamma_1}{\lambda_1 - \lambda_2 - \lambda_3} x_2 x_3 \right) e^{-(\lambda_1+\alpha_3\lambda_3)t} \tag{10}$$

where $\alpha_3 = -N_{13}/N_{33}$.

(ix) $\gamma_3 = 0$, $\lambda_i = \lambda$ $\forall i$, $d_{12}N_{11}N_{23} + d_{21}N_{22}N_{13} = 0$, $N_{11}N_{32} = N_{31}N_{22}$, $N_{33} = 0$ and $d_{21}\gamma_1 = d_{12}\gamma_2$ (implying det $N_{ij} = 0$ and $R_{123} = 0$):

$$I = x_3^{-2N_{22}/N_{32}} \left[x_1^2 - \frac{d_{12}}{d_{21}} x_2^2 - \frac{d_{13}}{d_{31}} x_3^2 \right] e^{-2(1-N_{22}/N_{32})\lambda t} . \tag{11}$$

Equation (11) coincides with (4), the LVS invariant II.

(x) $\gamma_3 = 0$, $\lambda_i = \lambda$ $\forall i$, $N_{13} = N_{33}$, $N_{23} = N_{33}$, $N_{31} = N_{11}$, $N_{32} = N_{22}$, and $d_{21}\gamma_1 = d_{12}\gamma_2$:

$$I = x_3^{-2} \left(x_1^2 - \frac{d_{12}}{d_{21}} x_2^2 \right) . \tag{12}$$

This invariant is similar to (4). In fact the same conditions in λ_i are shared and the conditions on N_{ij} here imply that $R_{123} = 0$.

(xi) $\gamma_3 = 0$, $N_{22}d_{21}\lambda_1 + N_{11}d_{12}\lambda_2 = 0$, $N_{13} = N_{33}$, $N_{23} = N_{33}$, $N_{31} = N_{11}$, $N_{32} = N_{22}$ and $d_{21}\gamma_1 = d_{12}\gamma_2$:

$$I = x_3^{-2} \left[1 + \frac{N_{11}}{\lambda_1} x_1^2 + \frac{N_{22}}{\lambda_2} x_2^2 + \frac{N_{33}}{\lambda_3} x_3^2 \right] e^{2\lambda_3 t} \tag{13}$$

which coincides with (6) taking the above conditions into account.

(xii) $\gamma_3 = 0$, $\lambda_1 = \lambda_2$, $\lambda_3 = 2\lambda_2$, $N_{23} = N_{13}$, $N_{31} = 2N_{11}$, $N_{32} = 2N_{22}$, $N_{33} = 2N_{13}$ and $d_{21}\gamma_1 = d_{12}\gamma_2$:

$$I = x_3^{-1} \left(x_1^2 - \frac{d_{12}}{d_{21}} x_2^2 \right) . \tag{14}$$

(xiii) $\gamma_3 = 0$, $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = N_{11} = N_{22} = N_{13} = N_{23} = 0$, and $N_{21}\gamma_1 = N_{12}\gamma_2$:

$$I = \left(x_1^2 - \frac{N_{12}}{N_{21}} x_2^2 \right) e^{-2\lambda t} . \tag{15}$$

(xiv) $\lambda_i = \lambda$, $N_{ii} = 0$ $\forall i$, $\gamma_1 - (N_{12}/N_{21})\gamma_2 - (N_{13}/N_{31})\gamma_3 = 0$ and $N_{12}N_{23}N_{31} + N_{21}N_{32}N_{13} = 0$:

$$I = \left(x_1^2 - \frac{N_{12}}{N_{21}} x_2^2 - \frac{N_{13}}{N_{31}} x_3^2 \right) e^{-2\lambda t} . \tag{16}$$

Invariants (v)–(viii) are for $\gamma_1 \neq 0$, and it is obvious that there exist two more corresponding sets of invariants and conditions by considering ($\gamma_1 = \gamma_2 = 0$) and ($\gamma_1 = \gamma_3 = 0$), and by changing the indices appropriately in the subsequent results. The first two of these were found by Goriely [6] using pseudo-monomial transformations. Invariants (ix)–(xiii) are similar to LVS ones but requiring more conditions among the system parameters. Two more invariants of the type (16) exist by the rotation ($1 \to 2, 2 \to 3, 3 \to 1$) of the indexes both in the invariant conditions and in the invariant. We note that $L, M, N = 1$ does not give any invariant in contradistinction with the LVS case. We also see that the fact of breaking the LVS structure, which results in considering the quadratic terms in (1), leads to an increase on the invariant conditions. In the course of calculations we encountered many trivial invariants which we choose not to present. The trivial invariants originate when the system (1), degenerates into one or two uncoupled equations. We also note the existence of invariants of LVS type (in this last case all three $\gamma_i = 0$) even though the system (1) does not adopt the form of a LVS system as two of the γ_is are not zero. To explain this fact we already noticed that the LVS invariant conditions were satisfied. Moreover, although the systems look different, their equilibrium points coincide. One can in fact see immediately that this is true in the plane $x_3 = 0$ for (11) or (12) and in the coordinate axis for (13).

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